Modelling musical chords using sine waves

Introduction

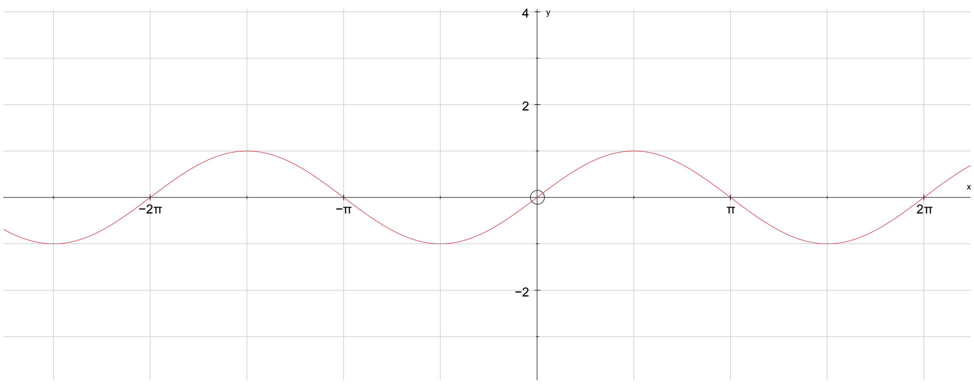
From the stimulus word "Harmony", I chose to look at the transmission of sound waves in music. As a keen musician myself, I was curious to understand more about how electronic instruments emit sound. Sound travels as a transverse wave, so takes the same basic shape as a mathematical sine wave. Electronic instruments use these waves to produce sounds of different frequencies, volume and timbre.

Oscillators[1] are used in most electronic instruments to produce sound waves. These sound waves are sine waves, with different frequencies, changing the pitch of the note, and different amplitudes, varying the volume of the note. The timbre of the sound is changed once the wave has left the oscillator. It travels through various filters which change the type of sound. This allows for a great range of sounds, that can imitate different instruments. It is this technology that is used in electronic keyboards.

I decided to focus on the changing of frequencies to produce different notes. By modelling chords, and series of notes using sine waves, I can demonstrate how the sound waves travel in the air, and how the different frequency waves relate to each other in different chords. I will look specifically at the relationships within each type of chord, and whether there is any difference between chords that are perceived to be in harmony and those that are perceived as discordant.

Investigation

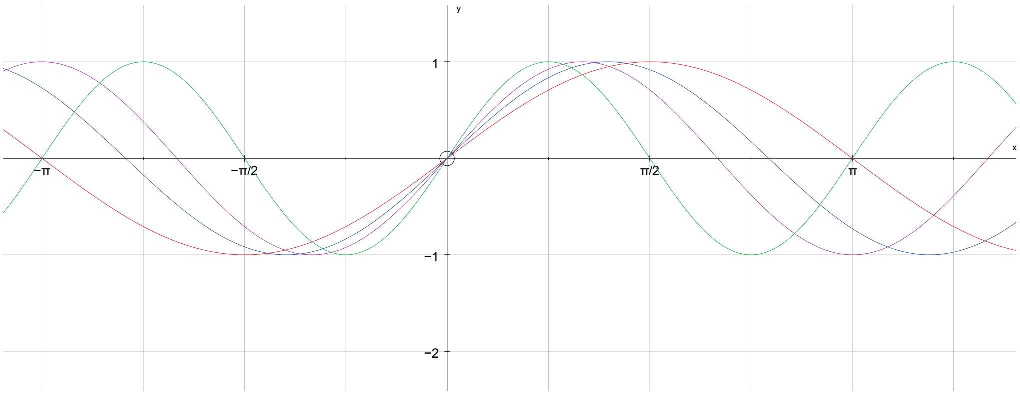
I started by working out how to model a single note. I decided to use "Middle" A, as it has an exact frequency of 440 Hz[2], and is the note that Western orchestras tune to. I wanted to make the sound wave of this note equal to the sine wave, which meant making 220Hz equivalent to$\pi\pi$. This produces the following graph:



[1]http://en.wikipedia.org/wiki/Synthesizer
[2] http://www.phy.mtu.edu/~suits/notefreqs.html

1. The Major Chord

Using an A major chord, I can look at the relationship between the notes in a major chord. The following graph demonstrates how this chord travels through air:



Equation 1: y=sinx
Equation 2: y=sin(554.37/440x)
Equation 3: y=sin(659.26/440x)
Equation 4: y=sin2x

I altered the wavelengths, or frequencies, of the graphs by changing the coefficient of x. In order to get the most accurate graph, I used the original frequencies of the other notes[3]. The table below shows more information about the notes in this chord.

Note	Semitones away from A	Frequency (Hz)	Approximate frequency ratio (Hz/440)
A	0	440.00	1
C#	4	554.37	1.25
E	7	659.26	1.50
A	12	880.00	2

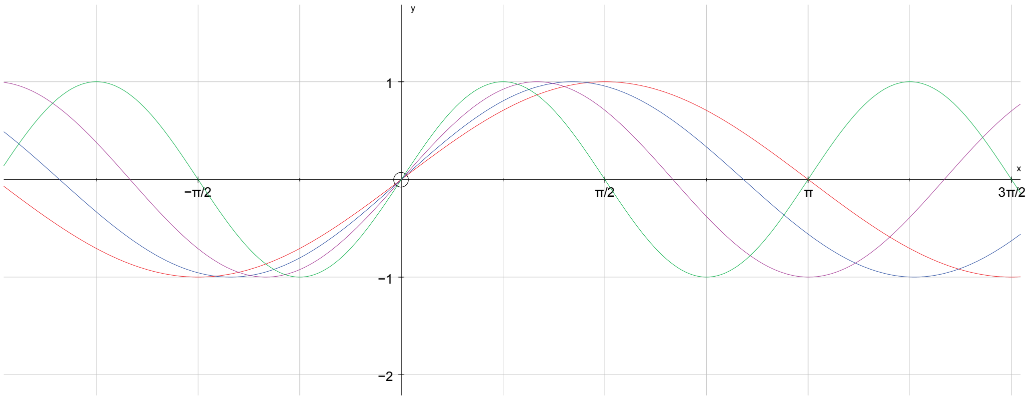
The frequency ratios in this chord appear to be neatly spaced, with the third gap in the chord equalling the sum of the first 2 gaps in the chord. The ratio of the gaps is 1:1:2. This may indicate the start of a pattern. Similarly, it could be that the third gap is double the first or second gap. I will now look at different chords to see if this conjecture holds

[3] http://www.phy.mtu.edu/~suits/notefreqs.html

true. The gaps in the ratios do not directly relate to the difference in semitones, as the semitones do not have equal frequencies-the frequencies of semitones diverge as the frequencies increase, and the notes get higher in pitch.

2. The Minor Chord

Minor chords are also considered to be in harmony. Again using an A chord, I have modelled a minor chord and looked at its features. These waves look very similar to the major chord waves, as only one note has changed; the C# has gone down by a semitone. Therefore the graphs are very similar. The graph for the minor chord looks like this:



Equation 1: y=sinx
Equation 2: y=sin(523.25/440x)
Equation 3: y=sin(659.26/440x)
Equation 4: y=sin2x

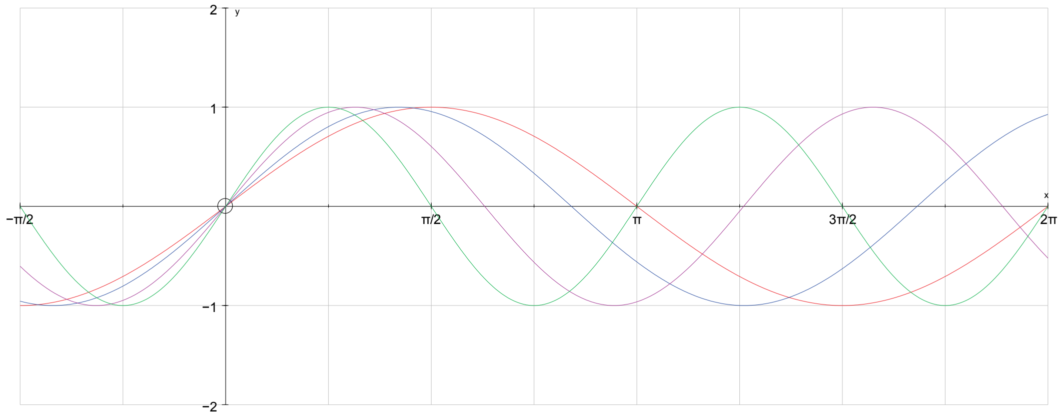
This graph looks very similar to the major graph, but the actual values, shown in the following table, are slightly different.

Note	Semitones away from A	Frequency (Hz)	Approximate frequency ratio (Hz/440)
A	0	440.00	1
C	3	523.25	1.20
E	7	659.26	1.50
A	12	880.00	2

Again, because the E in the middle of the chord has not moved, the sum of the first 2 gaps in the chord is equal to the third gap. The ratio of the gaps is now 2:3:5. However, to test this theory properly, we need to look at chords that differ from the original major chord more dramatically.

3. The First inversion major chord

An inversion chord is the same as a major chord, but the notes are in a slightly different order. A first inversion chord starting on A will actually be in the key of F major. The graph of such a chord looks like this:



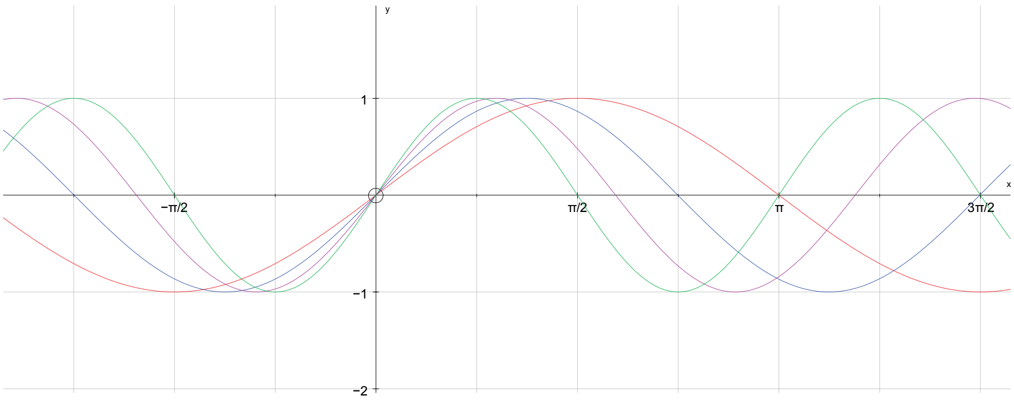
Equation 1: y=sinx
Equation 2: y=sin(523.25/440x)
Equation 3: y=sin(698.46/440x)
Equation 4: y=sin2x

Note	Semitones away from A	Frequency (Hz)	Approximate frequency ratio (Hz/440)
A	0	440.00	1
C	3	523.25	1.20
F	8	659.26	1.60
A	12	880.00	2

The table of values for this chord shows that my original conjecture does not hold true for all chords. However, the gaps are still in a neat ratio of 1:2:2.

4. The Second inversion major chord

This chord, like the first inversion, is another different rearrangement of the notes in a major chord.



Equation 1: y=sinx
Equation 2: y=sin(587.33/440x)
Equation 3: y=sin(739.99/440x)
Equation 4: y=sin2x

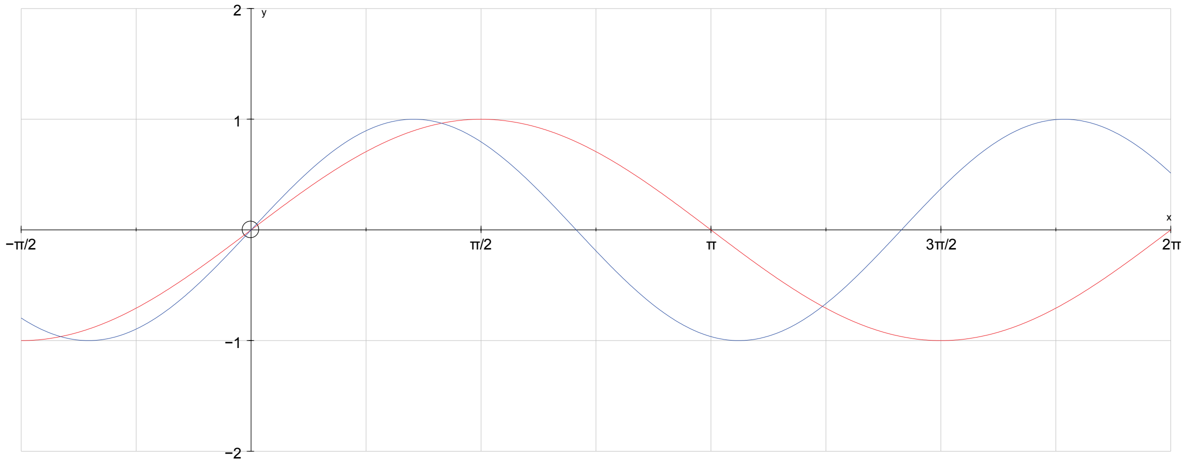
Note	Semitones away from A	Frequency (Hz)	Approximate frequency ratio (Hz/440)
A	0	440.00	1
D	5	587.33	1.33
F#	9	739.99	1.67
A	12	880.00	2

Interestingly, in this chord, the frequencies of the notes are evenly spaced, as the ratio of the gaps is 1:1:1.

I will now look at some discordant notes, and see how they compare with the chords I have already looked at.

5. The Augmented 4^{th}

This chord is also known as "The Devil's Chord", as it is considered to be the most unpleasant sounding chord in music. As it is only 2 notes, it cannot be modelled in exactly the same way as the other chords, but can still be graphed:



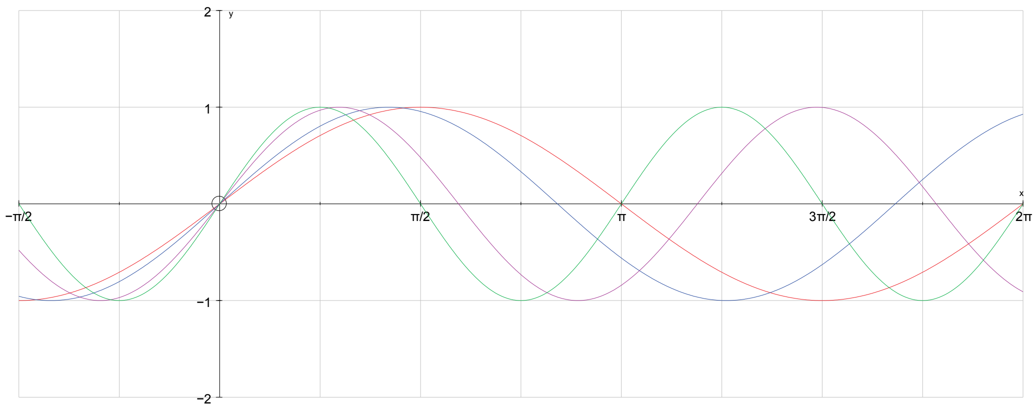
Equation 1: y=sinx
Equation 2: y=sin(622.25/440x)

Note	Semitones away from A	Frequency (Hz)	Approximate frequency ratio (Hz/440)
A	0	440.00	1
D#	6	622.25	1.41

The approximate ratio for this chord shows that the ratio is not as precise as with the other chords. However, there is not necessarily a difference in chords that are in harmony and discordant chords. This chord is not directly comparable to the other chords, as it does not have 3 notes like the others.

6. The Augmented 6^{th}

I will now look at the augmented 6^{th}, another discordant chord. This chord however, has 3 notes, so can be more easily compared to the other chords I have looked at. An Augmented 6^{th} is also known as a dominant 7^{th}, which was one of the first discordant chords to be used in music in the West.



Equation 1: y=sinx
Equation 2: y=sin(523.25/440x)
Equation 3: y=sin(739.99/440x)
Equation 4: y=sin2x

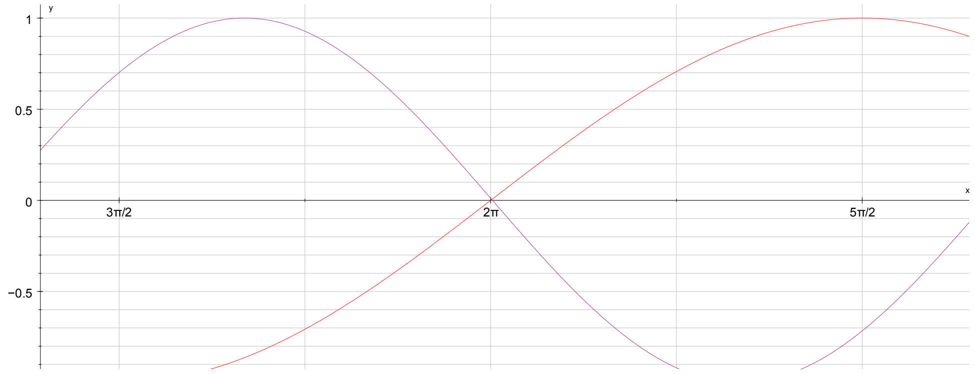
Note	Semitones away from A	Frequency (Hz)	Approximate frequency ratio (Hz/440)
A	0	440.00	1
C	3	523.25	1.2
F#	9	739.99	1.67
A	12	880.00	2

In this chord, the gaps in the ratios are not easily relatable to each other, suggesting that discordant sets of notes are not as natural as those that sound harmonic.

<u>"Perfect Intervals"</u>

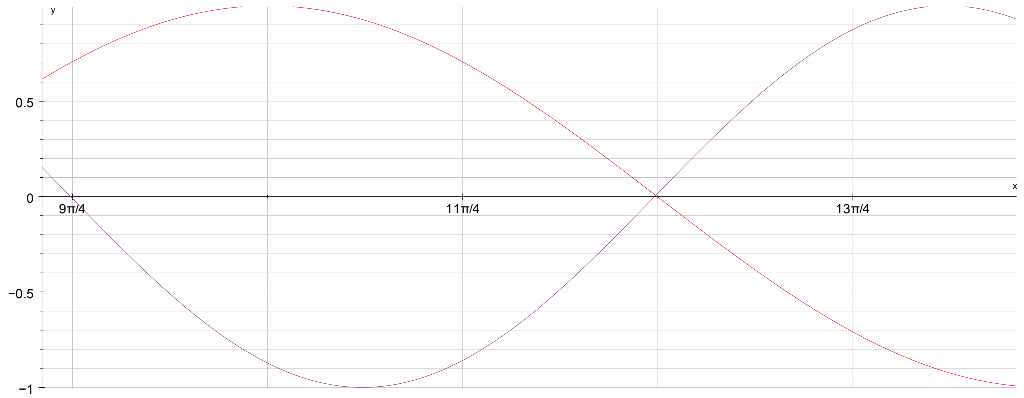
The intervals known as a "perfect fourth" and "perfect fifth" are, as the name suggests, considered to be the most harmonic intervals. Interestingly, the graphs of these intervals show that the notes in the perfect fourth cross at exactly 3π, and the notes in the perfect fifth cross at 2π.

The graph of a perfect fourth:



Equation 1: y=sinx
Equation 2: y=sin(587.33/440x)

The graph of a perfect fifth:



Equation 1: y=sinx
Equation 2: y=sin(659.26/440x)

Conclusion

This evidence would suggest that notes which are traditionally considered to be "in harmony" are mathematically "neater" than those that are not in harmony. The mathematical nature of the notes in traditional harmony suggests that the sound waves are more pleasing to our brain because of the way the notes fit together. The ratios of frequencies of chords in harmony are more concise than the discordant notes. The sound waves produced by oscillators are in the same form as notes produced by mechanical instruments, so the same note ultimately sounds the same, or very similar, to our ear. For this reason, this theory does not only apply to sound coming to electronic instruments, but every kind of instrument.

Evaluation

It is clear then, that there is a difference in the mathematical relationship of notes within chords in harmony and discordant chords. This means, that perhaps in the future music can be created entirely mathematically. It may be that maths can help determine which chords, and sequences of chords, will work together and be most pleasing to the ear.

This investigation was very limited, as I was not able to research properly how electronic instruments work, and how they differ from mechanical instruments. I also would have liked to tried to find a mathematical formula connecting the number of semitones to the frequency ratios. The evidence here suggests that notes in harmony fit into a mathematical context, whereas discordant notes do not. However, this is just a theory, and would have to be tested more thoroughly. To extend this investigation, I would like to look at the area between the waves in relation to the semitone difference and frequency ratios.

Bibliography

Suits, B.H., 2010. *Frequencies of musical notes* [Website] Available at: http://www.phy.mtu.edu/~suits/notefreqs.html [Accessed: 23rd June 2010]

Wikipedia *Synthesiser*. Anon [Website] Available at: http://en.wikipedia.org/wiki/Synthesizer [Accessed: 23rd June 2010]